DEFINITION Derivative of a Function

The derivative of the function \( f \) with respect to the variable \( x \) is the function \( f' \) whose value at \( x \) is

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

provided the limit exists.

You will want to recognize this formula (a slope) and know that you need to take the derivative of \( f(x) \) when you are asked to find \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).
 DEFINITION (ALTERNATE)  Derivative at a Point

The derivative of the function \( f \) at the point \( x = a \) is the limit

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

provided the limit exists.

This is the slope of a segment connecting two points that are very close together.
DEFINITION  Continuity

A function $f$ is continuous at a number $a$ if

1) $f(a)$ is defined ($a$ is in the domain of $f$)

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

A function is continuous at an $x$ if the function has a value at that $x$, the function has a limit at that $x$, and the value and the limit are the same.

Example:

Given $f(x) = \begin{cases} x^2 + 3, & x \leq 2 \\ 3x + 2, & x > 2 \end{cases}$

Is the function continuous at $x = 2$?

$f(x) = 7$

$\lim_{x \to 2^-} f(x) = 7$, but the $\lim_{x \to 2^+} f(x) = 8$

The function does not have a limit as $x \to 2$, therefore the function is not continuous at $x = 2$. 
Limits as $x$ approaches $\infty$

For rational functions, examine the $x$ with the largest exponent, numerator and denominator. The $x$ with the largest exponent will carry the weight of the function.

If the $x$ with the largest exponent is in the denominator, the denominator is growing faster as $x \to \infty$. Therefore, the limit is 0.

$$\lim_{x \to \infty} \frac{3 + x}{x^4 - 3x + 7} = 0$$

If the $x$ with the largest exponent is in the numerator, the numerator is growing faster as $x \to \infty$. The function behaves like the resulting function when you divide the $x$ with the largest exponent in the numerator by the $x$ with the largest exponent in the denominator.

$$\lim_{x \to \infty} \frac{3 + x^5}{x^2 - 3x + 7} = \infty$$

This function has end behavior like $x^3 \left(\frac{x^5}{x^2}\right)$.

To say the limit equals infinity gives a very good picture of the behavior.

If the $x$ with the largest exponent is the same, numerator and denominator, the limit is the coefficients of the two $x$’s with that largest exponent.

$$\lim_{x \to \infty} \frac{3 + 4x^5}{7x^5 - 3x + 7} = \frac{4}{7}$$  As $x \to \infty$, those $x^5$ terms are like gymnasiums full of sand.

The few grains of sand in the rest of the function do not greatly affect the behavior of the function as $x \to \infty$. 
LIMITS

\[ \lim_{{x \to c}} f(x) = L \]

The limit of \( f \) of \( x \) as \( x \) approaches \( c \) equals \( L \).

As \( x \) gets closer and closer to some number \( c \) (but does not equal \( c \)), the value of the function gets closer and closer (and may equal) some value \( L \).

One-sided Limits

\[ \lim_{{x \to c^-}} f(x) = L \]

The limit of \( f \) of \( x \) as \( x \) approaches \( c \) from the left equals \( L \).

\[ \lim_{{x \to c^+}} f(x) = L \]

The limit of \( f \) of \( x \) as \( x \) approaches \( c \) from the right equals \( L \).

Using the graph above, evaluate the following:

\[ \lim_{{x \to 1^-}} f(x) = \]
\[ \lim_{{x \to 1^+}} f(x) = \]
\[ \lim_{{x \to 1}} f(x) = \]
Practice Problems

Limit as $x$ approaches infinity

1. \[ \lim_{{x \to \infty}} \left( \frac{3x - 7}{5x^4 - 8x + 12} \right) = \]

2. \[ \lim_{{x \to \infty}} \left( \frac{3x^4 - 2}{5x^4 - 2x + 1} \right) = \]

3. \[ \lim_{{x \to \infty}} \left( \frac{x^6 - 2}{10x^3 - 9x + 8} \right) = \]

4. \[ \lim_{{x \to \infty}} \left( \frac{7x^4 - 2}{5 - 2x^3 - 14x^4} \right) = \]

5. \[ \lim_{{x \to \infty}} \left( \frac{\sin x}{e^x} \right) = \]

6. \[ \lim_{{x \to -\infty}} \left( \frac{\sqrt{x^2 - 9}}{2x - 3} \right) = \]

7. \[ \lim_{{x \to \infty}} \left( \frac{\sqrt{x^2 - 9}}{2x - 3} \right) = \]
Practice Problems

Limit as $x$ approaches a number

8. $\lim_{x \to 2} (x^3 - x + 1)$

9. $\lim_{x \to 2} \left( \frac{x^2 - 4}{x - 2} \right) = \ldots$

10. $\lim_{x \to 2} \left( \frac{3}{x - 2} \right) = \ldots$

11. $\lim_{x \to 2} \left( \frac{3}{x - 2} \right) = \ldots$

12. $\lim_{x \to 2} \left( \frac{3}{x - 2} \right) = \ldots$

13. $\lim_{x \to 2} \left( \frac{3}{2 - x} \right) = \ldots$

14. $\lim_{x \to \frac{\pi}{4}} \left( \frac{\sin x}{x} \right) = \ldots$

15. $\lim_{x \to \frac{\pi}{4}} \left( \frac{\tan x}{x} \right) = \ldots$
1. What is \( \lim_{{h \to 0}} \frac{\sin(x + h) - \sin(x)}{h} \) ?

(A) \( \sin x \)  (B) \( \cos x \)  (C) \(-\sin x\)

(D) \(-\cos x \)  (E) The limit does not exist

2. \( \lim_{{\Delta x \to 0}} \frac{\cos \left( \frac{\pi}{3} + \Delta x \right) - \cos \left( \frac{\pi}{3} \right)}{\Delta x} = \)

(A) \( -\frac{\sqrt{3}}{2} \)  (B) \( -\frac{1}{2} \)  (C) 0

(D) \( \frac{1}{2} \)  (E) \( \frac{\sqrt{3}}{2} \)

3. \( \lim_{{h \to 0}} \frac{(x + h)^3 - x^3}{h} = \)

(A) \(-x^3 \)  (B) \(-3x^2 \)  (C) \(3x^2 \)

(D) \(x^3 \)  (E) The limit does not exist
4. The graph of \( y = f(x) \) is shown above. \( \lim_{x \to 2} \left( (f(x))^3 - 3f(x) + 7 \right) = \)

(A) 1  (B) 5  (C) 7  (D) 9  (E) Does not exist

5. If \( f(x) = \begin{cases} x^2 - 3x - 4, & x \neq -1 \\ 2, & x = -1 \end{cases} \), what is \( \lim_{x \to -1} f(x) \)?

(A) -5  (B) 0  (C) 2  (D) 3  (E) Does not exist

6. \( \lim_{x \to \infty} \left( \frac{2x^6 - 5x^3 + 10}{20 - 4x^2 - x^6} \right) = \)

(A) -2  (B) -\frac{1}{2}  (C) \frac{1}{2}  (D) 2  (E) Does not exist

7. \( \lim_{x \to \infty} \left( \frac{2x^5 - 5x^3 + 10}{20 - 4x^2 - x^6} \right) = \)

(A) -2  (B) -\frac{1}{2}  (C) 0  (D) \frac{1}{2}  (E) 2
8. \[ \lim_{{x \to \infty}} \left( 1 + e^{\frac{1}{x^2}} \right) = \]

(A) \(-\infty\)  (B) 0  (C) \(e^{\frac{1}{2}}\)

(D) \(1 + e^{\frac{1}{2}}\)  (E) \(\infty\)

9. \[ \lim_{{x \to 3^-}} \frac{5}{3 - x} = \]

(A) \(-\infty\)  (B) \(-5\)  (C) 0

(D) \(\frac{5}{3}\)  (E) \(\infty\)

10. If \( \lim_{{x \to \infty}} \left( \frac{5n^3}{20 - 3n - kn^3} \right) = \frac{1}{2} \), then \(k = \)

(A) \(-10\)  (B) \(-4\)  (C) \(\frac{1}{4}\)  (D) 4  (E) 10

11. Which of the following is/are true about the function \(g\) if \(g(x) = \frac{(x-2)^2}{x^2 + x - 6}\)?

I. \(g\) is continuous at \(x = 2\)
II. The graph of \(g\) has a vertical asymptote at \(x = -3\)
III. The graph of \(g\) has a horizontal asymptote at \(y = 0\)

(A) I only  (B) II only  (C) III only  (D) I and II only  (E) II and III only
12. \( f(x) = \begin{cases} 
\sin x, & x < \frac{\pi}{4} \\
\cos x, & x > \frac{\pi}{4} \\
\tan x, & x = \frac{\pi}{4} 
\end{cases} \)

What is \( \lim_{x \to \frac{\pi}{4}} f(x) \)?

(A) \(-\infty\)  (B) 0  (C) 1  (D) \(\frac{\sqrt{2}}{2}\)  (E) \(\infty\)

13. \( \lim_{x \to a} \left( \frac{\sqrt{x} - \sqrt{a}}{x - a} \right) = \)

(A) \(\frac{1}{2\sqrt{a}}\)  (B) \(\frac{1}{\sqrt{a}}\)  (C) \(\sqrt{a}\)  (D) \(2\sqrt{a}\)  (E) Does not exist

14. \( \lim_{x \to 0} \frac{\ln 2x}{2x} = \)

(A) \(-\infty\)  (B) \(-1\)  (C) 0  (D) 1  (E) \(\infty\)

15. At \( x = 4 \), the function given by \( h(x) = \begin{cases} 
x^2, & x \leq 4 \\
4x, & x > 4 
\end{cases} \) is

(A) Undefined  
(B) Continuous but not differentiable  
(C) Differentiable but not continuous  
(D) Neither continuous nor differentiable  
(E) Both continuous and differentiable
Free Response 1

Let $h$ be the function defined by the following:

$$h(x) = \begin{cases} 
|x-1| + 3, & 1 \leq x \leq 2 \\
ax^2 - bx, & x > 2
\end{cases}$$

$a$ and $b$ are constants.

(a) If $a = -1$ and $b = -4$, is $h(x)$ continuous for all $x$ in $[1, \infty]$? Justify your answer.

(b) Describe all values of $a$ and $b$ such that $h$ is a continuous function over the interval $[1, \infty]$.

(c) The function $h$ will be continuous and differentiable over the interval $[1, \infty]$ for which values of $a$ and $b$?
Free Response 2 (No calculator)

Given the function \( f(x) = \frac{x^3 + 2x^2 - 3x}{3x^2 + 3x - 6} \).

(a) What are the zeros of \( f(x) \)?

(b) What are the vertical asymptotes of \( f(x) \)?

(c) The end behavior model of \( f(x) \) is the function \( g(x) \). What is \( g(x) \)?

(d) What is \( \lim_{x \to \infty} f(x) \)? What is \( \lim_{x \to \infty} g(x) \)?